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OPTIMIZING OPINION INFLUENCE IN MULTI-ZONE CONTEXT: STRATEGIES IN AGREE-DISAGREE MODELS

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Abstract. In this paper, we aim to investigate optimal control to a mathematical model that describes agree-disagree opinions during polls in a multi-zone framework. We first present the model and recall its different compartments and we present some properties of the multi-zone model. We formulate the optimal control problem by supplementing our model with a functional objective. Optimal control strategies are proposed to reduce the number of disagreeing people in a targeted zone and the cost of interventions. We prove the existence of solutions to the control problem, and we employ the Pontryagin's maximum principle to find the necessary conditions for the existence of the optimal controls and Runge- Kutta forward-backward sweep numerical approximation method is used to solve the optimal control system, and perform numerical simulations using various initial conditions and parameters to investigate several scenarios.

Keywords: Opinion control, optimal control, perception spread, network analysis, statistical analysis.
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1 Introduction

Public opinion is like a collective voice, reflecting what different groups in society feel about real-life events that impact their interests (Mut, 1998). It's a way for people to express their thoughts openly, influencing how society works and the political system (Dalton, 2013). In democratic societies, getting more involved in politics means individuals can freely choose whether to support those in power or oppose them by joining opposition groups (Soroka & Wlezien, 2010). It's a personal choice on a democratic stage.

When it comes to elections, public opinion polls have several important roles. They work as a form of direct voting, showing what people think. They also help gather information for shaping election strategies, and they even become part of the election process itself, where ideas are presented and debated (Milburn, 1991). The engagement of citizens in politics and their level of participation depend on three key factors: Firstly, it involves maintaining the activity levels of citizens who support the government and its policies, while also striving to increase their involvement. Secondly, it requires drawing politically inactive citizens into the ranks of

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active participants. Lastly, it encompasses the active and confrontational stance of opposition leaders, while also neutralizing the actions of ordinary participants and clarifying the true goals of state policy that need to be pursued (Marinetto, 2003).

Membership in a political group holds little value if its members cannot be motivated to actively and effectively support their party's objectives (Scarrow, 1994). However, achieving this demands significant time and financial investment ([Bidah et al., 2020b). Manipulating public opinion during elections provides politicians with the means to make strategic decisions (Lust-Okar, 2004). Public opinion serves as a gauge of the collective sentiment surrounding specific issues or political figures (Kim et al., 1999), enabling citizens to be engaged in political processes (Gilens & Page, 2014). However, individuals tend to overestimate their opinions, particularly in political elections, where they assert a shared standpoint. This tendency toward exaggeration can create negative perceptions. It's important to note that public opinion represents the entirety of opinions, regardless of their varying degrees of certainty ([Bidah et al., 2020b).

Elections serve as a reflection of public sentiment. Under universal suffrage, the "sample" encompasses nearly all eligible adult citizens who wish to select their governing representative. This involves interpreting election outcomes and understanding voting motivations. Referendums and plebiscites are also tools for gauging public opinion, extending to a wider array of matters. In international law, a plebiscite often equates to a referendum. A broad view of "plebiscite" suggests any vote revealing voter opinion on a specific issue (Smith, 2000; Rokkan, 2009).

Ultimately, the authority to make decisions rests with the state or senior officials. Plebiscite democracy is no exception; leaders may ascribe imperative weight to citizen opinions expressed through a popular vote. In referendums, the electorate directly decides certain political matters, while plebiscites entail public input on choices made by the government or head of state. Interestingly, the influence of plebiscite results on the government mirrors the impact of sociological measurements. By capturing citizen stances on particular issues, surveys bear a resemblance to voting (Jackman, 2005; Patterson, 2005).

Public opinion polls play a crucial role in comprehending voter sentiment and preferences, which in turn shapes the strategies and tactics of political entities during election campaigns (Morwitz & Pluzinski, 1996). Their importance is evident in the mounting interest they garner from governmental bodies, private organizations, candidates, and research funds focused on political sociology. This has led to an extensive body of work dedicated to these issues. Voters employ such polls to uphold or attain cognitive consistency, based on their expectations for winning candidates and their intended votes (Jacobs & Shapiro, 1995).

The increasing significance of public opinion polls is illustrated by rising political party expenditures on them, particularly during election seasons (Doob, 1948). The United States leads in this domain, boasting over 200 specialized firms that have refined the art of "measuring" public sentiment to an advanced level (Harris, 1963). Globally, public opinion polls have gained prominence, although not all countries have embraced them with the depth and precision found in the United States. Nevertheless, they are increasingly adopted worldwide as vital tools for gauging public sentiments (Marsh, 1985). For instance, France hosts around 150 organizations with about 10 thousand employees engaged in public opinion research. Similar organizations and services have also emerged in Russia (Gallup, 1976).

Public opinion polls function as a form of political intelligence, shedding light on public stances regarding various political matters. They're designed to ascertain voters' opinions about specific political issues, concerns, as well as the effectiveness of domestic and foreign political actions by the government (Manza et al., 2002). Over time, these polls have evolved from mere mood indicators to tools that guide those very sentiments. They facilitate the identification, organization, and publication of public opinions without requiring action from opinion holders (Gallup & Newport, 2006).

However, polls also possess the potential for steering public opinion in directions less detri-

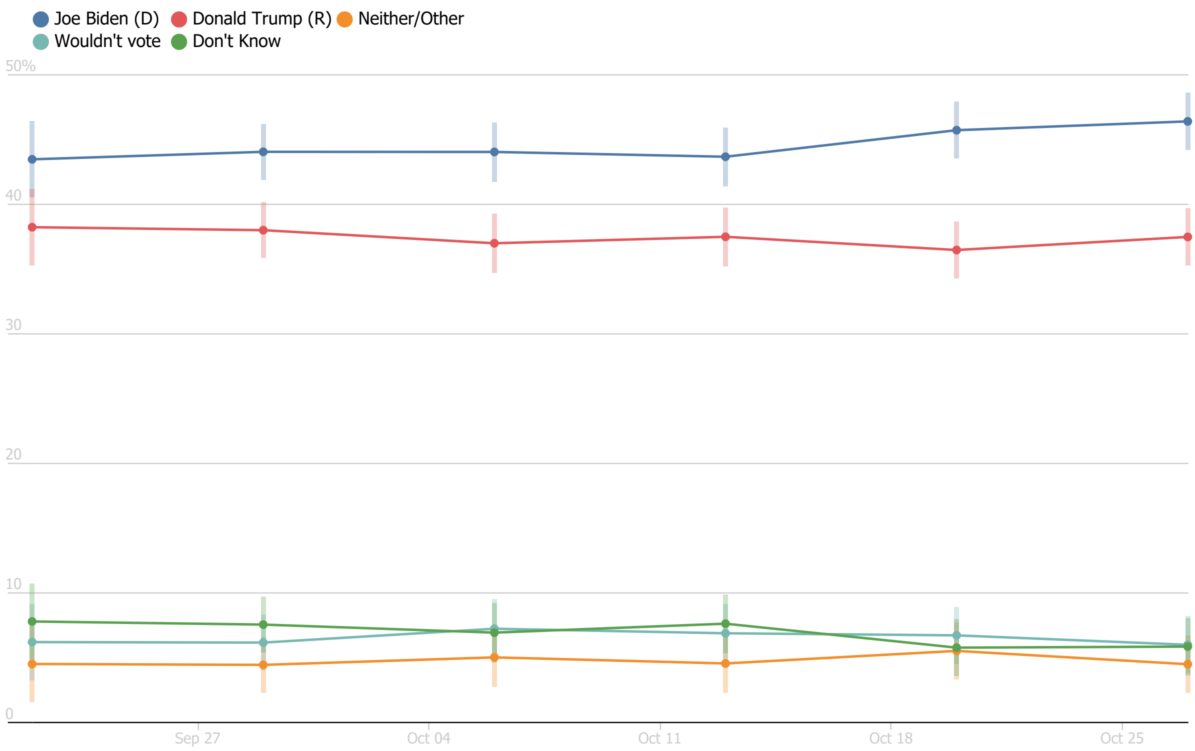

Figure 1: Public opinion poll in the Unite states of America carried out by the Reuters polling system regarding the presidential election Reuters polling system (2020a): Data from 2020-09-22 to 2020-10-27. Poll Question: If the 2020 presidential election were held today, how would you vote?

mental to the current regime. They often contain an element of programming and can be employed to manipulate public sentiment. Therefore, caution is necessary when interpreting poll data meant to reveal the ideological positions of different population segments, their assessments of political party programs, and individual political figures. The results of widely covered polls have a direct impact on the tone and substance of election campaigns, compelling candidates to adjust their positions, align with certain social groups, and impact their popularity among voters (Gallup & Rae, 1940; Gallup & Newport, 2006).

Mathematical models serve a pivotal role in elucidating real-world phenomena. In their work ([Bidah et al., 2020c), the authors introduced and analyzed a novel mathematical model that specifies the dynamics of agree-disagree opinions within the context of polls. Their efforts encompassed a series of computational and statistical experiments that substantiated their theoretical findings. Moreover, they delved into the influence of pivotal model parameters on equilibrium thresholds, unearthing valuable insights. Meanwhile, the work in [Bidah et al. (2020b) took a statistical approach, scrutinizing data from the Reuters polling system to explore the approval ratings of U.S. presidential terms.

In a similar vein, the authors' work Bidah et al. (2020a) explores the problem of optimal control, anchored in the agree-disagree model in one region. This exploration forms the background for our current study. In this paper, we unveil a generalized mathematical model, that goes beyond individual viewpoints by including amplification of opinions across diverse geographic regions. This new view allows for an enhanced understanding of the probabilities of outcomes. In contrast to the work in Bidah et al. (2020a), where all studied zones were treated as a single entity without accounting for crucial geographical and cultural parameters. Knowing that these factors intricately shape the evolution of opinions, adding layers of complexity to the dynamics. Our study takes into account the profound influence of geographical nuances, cultural diversity, and other pertinent variables that play pivotal roles in opinion dynamics. This is an important distinction that marks the novelty of our work, by considering the multifaceted impact of these variables, we contribute a comprehensive perspective that more accurately mirrors the intricacies of real-world opinion evolution across diverse geographic zones

In this article, a generalized mathematical model of the one studied in Bidah et al. (2020a), is presented to define how opinions are increased during polls in different geographic zones which facilitates the anticipation of probability of result. We start by presenting the multi-zone mathematical model that describes the evolution of opinions during public opinion polls in several regions and some mathematical properties. Then, we introduce two control functions in a targeted zone that represent the positive and negative effects of the media and the publicity on the people's attitudes. We prove mathematically the existence of these controls, and we characterize the optimal controls aiming to minimize an objective function using the Pontryagin's Maximum Principle (Pontryagin et al., 1962; Mahmudov, 2021).

The paper is organized as follows: Section 2 introduces the multi-zone model, giving some details about the different compartments and parameters of the model and some mathematical properties. In section 3, we present the optimal control problem and we derive the sufficient conditions for the existence of controls and the necessary optimality conditions. Section 4 provides numerical results and discussion of several scenarios. Section 5 concludes.

2 Presentation of the model

In this section we introduce an extension to the mathematical model outlined in Bidah et al. (2020a), encompassing the progression of Agree and Disagree viewpoints within a multi-zone structure during Polls. The surveys under consideration involve responses that span agreement, disagreement, or other options pertaining to specific candidates or ideas. Additionally, we elaborate on the political stance of candidate parties, accommodating scenarios involving multiple parties while still maintaining the essence of decision-making within a binary framework.

We suppose that the studied domain Ω is composed of p geographical zones, denoted as C_j for $j = 1, ..., p$. The population of each zone, C_j, has the potential to interact with populations from other zones through various means like the internet or face-to-face communication. It follows that Ω can be represented as the union of all C_j zones.

As a consequence, the population within each zone, C_j, can be categorized into three distinct groups, and the model is structured accordingly into compartments. For every C_j, we delineate three compartments as follows:

- Indifferent Individuals (I^j): This group comprises individuals who are undecided, ambivalent, or unaware of the ongoing poll. They might also abstain from voting due to personal reasons. Their attitudes toward the ideas, parties, or candidates are weak or absent, lacking strong positive or negative associations.
- Agree Group (A^j): This category encapsulates individuals who express agreement with the subject of study, whether it be ideas, parties, or candidates.
- Disagree Group (D^j): Individuals falling into this group signify their disagreement with the matter under scrutiny.

The model's framework incorporates a set of hypotheses. The first assumption pertains to the distribution of the studied population, leading to a uniform dispersion of indifferent individuals throughout the entire populace. Secondly, the model factors out the influence of mortality and recruitment during the poll's course. The third supposition acknowledges the prospect of intercommunication within the population, implying that individuals engage in efforts to persuade one another. Lastly, instances of uncertainty in forming an opinion or instances of refusing to vote are construed as expressions of indifference.

Illustratively, Fig. 1 showcases an opinion poll conducted by the Reuters polling system from September 22, 2020, to October 27, 2020, focusing on the U.S presidential election. This poll

Table 1: Parameters descriptions and values.

Parameter	Description
β_1^{jk}	Indifferent to Agree transmission rate in C_j, due to contacts with agreeing people of C_k
β_2^{jk}	Indifferent to Disagree transmission rate in C_j, due to contacts with disagreeing people of C_k
α_1^{jk}	Disagree to Agree transmission rate in C_j, due to contacts with agreeing people of C_k
α_2^{jk}	Agree to Disagree transmission rate in C_j, due to contacts with disagreeing people of C_k
γ_1^j	Interest loss factor of Agree individuals of C_j
γ_2^j	Interest loss factor of Disagree individuals of C_j

revolves around the question: "If the 2020 presidential election were held today, how would you vote?" The graphical representation demonstrates varied responses, encompassing preferences for Joe Biden, Donald Trump, neither/other, abstention, and uncertainty. In the context of analyzing Joe Biden's political position, votes cast for him are categorized as agree opinions, those for Trump are interpreted as disagree opinions, while all other responses are regarded as expressions of indifference.

Individuals each hold unique rationales for aligning with or dissenting from a viewpoint. Within zone C_j, an indifferent individual's stance can be swayed toward agreement by individuals from differing zones, denoted C_k, who hold an agreeing standpoint. This persuasion unfolds at a rate symbolized as β_1^{jk}. Similarly, a contrary persuasion from disagreeing individuals at a rate β_2^{jk} can cause the same indifferent individual to shift their stance to disagreement.

Conversely, within the same framework, individuals within C_j who express agreement can be influenced to disagree by individuals from zone C_k at a rate α_2^{jk}. Likewise, individuals who hold dissenting opinions within C_j can be swayed to agree by individuals from C_k at a rate α_1^{jk}.

Furthermore, the realm of influence extends to scenarios where individuals might choose to abstain from voting or gradually lose interest without engaging in direct contact with individuals who hold contrasting opinions. In such instances, individuals aligned with agreement within C_j shift to an indifferent stance at a rate represented by γ_1^j. Correspondingly, individuals with dissenting views within C_j transition to an indifferent stance at a rate denoted as γ_2^j. Notably, all interactions adhere to a standard incidence rate.

The compilation of these assumptions and considerations is succinctly encapsulated in a system of ordinary differential equations, systematically delineating the dynamics within zone C_j:

$$I^{j\prime} = -\sum_{k=1}^{p} \beta_1^{jk} \frac{A^k I^j}{N_j} - \sum_{k=1}^{p} \beta_2^{jk} \frac{D^k I^j}{N_j} + \gamma_1^j A^j + \gamma_2^j D^j \tag{1}$$

$$A^{j\prime} = \sum_{k=1}^{p} \beta_1^{jk} \frac{A^k I^j}{N_j} + \sum_{k=1}^{p} \alpha_1^{jk} \frac{A^k D^j}{N_j} - \sum_{k=1}^{p} \alpha_2^{jk} \frac{A^j D^k}{N_j} - \gamma_1^j A^j \tag{2}$$

$$D^{j\prime} = \sum_{k=1}^{p} \beta_2^{jk} \frac{D^k I^j}{N_j} + \sum_{k=1}^{p} \alpha_2^{jk} \frac{A^j D^k}{N_j} - \sum_{k=1}^{p} \alpha_1^{jk} \frac{A^k D^j}{N_j} - \gamma_2^j D^j \tag{3}$$

Where $I^j(0) \geq 0$, $A^j(0) \geq 0$, and $D^j(0) \geq 0$. And $N_j = I^j + A^j + D^j$, note that $N_j{}' = I^{j\prime} + A^{j\prime} + D^{j\prime} = 0$, thus, the population size N_j of each zone C_j is considered as a constant in time. A summary of parameters description is given in Table 1.

We can easily prove that for non-negative initial conditions, the solutions of system (1)-(3) are non-negative. To do this, recall that by Mailleret (2004) the system of equation

$$x' = f(x_1, x_2, ..., x_k)$$

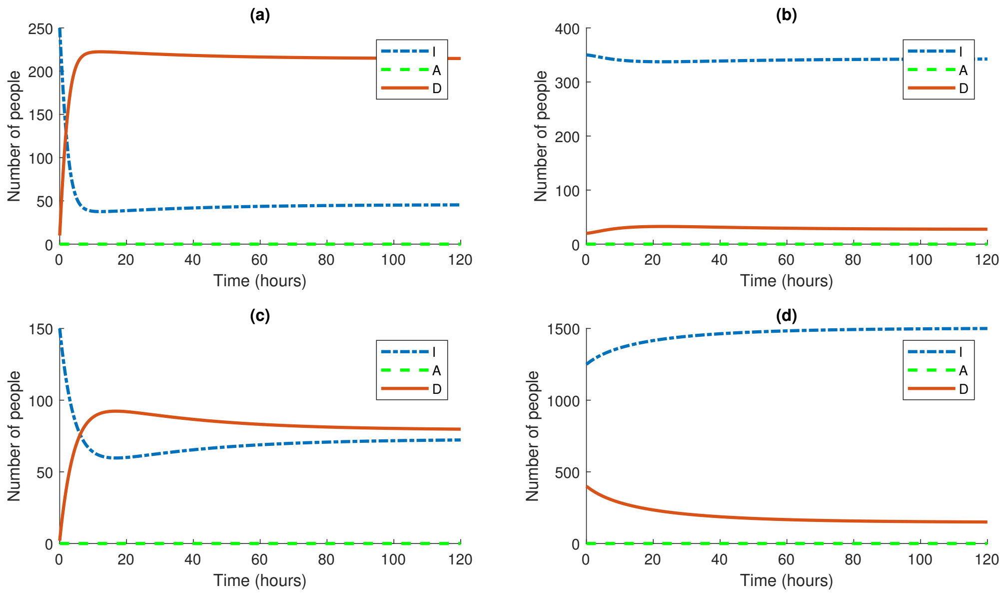

Figure 2: Example of the agree-free equilibrium discussed in Theorem (1).

with

$$x(0) = x_0 \geq 0$$

is a positive system if and only if, $\forall i = 1, 2, ..., k$

$$x_i' = f_i(x_1 \geq 0, x_2 \geq 0, ..., x_i = 0, ..., x_k \geq 0) \geq 0$$

Thus, for the model (1)-(3) it is easy to verify that for all j

$$\begin{aligned} I^j &= 0 \Rightarrow {I^j}' \geq 0 \\ A^j &= 0 \Rightarrow {A^j}' \geq 0 \\ D^j &= 0 \Rightarrow {D^j}' \geq 0 \end{aligned}$$

Therefore, all the solutions of system (1)-(3) are non-negative.

It is clear also that the solutions of the model (1)-(3) are bounded based on the fact that $N_j = I^j + A^j + D^j$ is constant, then $S^j \leq N^j$, $A^j \leq N^j$,and $D^j \leq N^j$. Therefore, we will focus to study the model (1)-(3) in the closed positively invariant feasible set, given by

$$\Omega = \left\{ \left(I^j, A^j, D^j\right) \in \mathbb{R}^3_+ \text{ for all } j / \, I^j + A^j + D^j \leq p \max_{j=1,...p} N_j \right\}$$

Definition 1. *The model* (1)-(3) *is said to be at an agree-free equilibrium for a zone j if $A^j = 0$ and ${A^j}' = 0$.*

The model (1)-(3) *is said to be at a disagree-free equilibrium for a zone j if $D^j = 0$ and ${D^j}' = 0$.*

Definition 2. *The model* (1)-(3) *is said to be at the agree-free equilibrium if it is at the agree-free equilibrium for all zones.*

The model (1)-(3) *is said to be at the disagree-free equilibrium if it is at the disagree-free equilibrium for all zones.*

Theorem 1. *Suppose that system* (1)-(3) *is at an agree-free equilibrium for some zone j. Then the model is at the agree free equilibrium i.e. $A^k = 0$ for each zone k.*

Proof. For simplicity suppose that $j = 1$, i.e. the zone 1 is at the agree-free equilibrium and there is no agree opinion in the zone 1, thus $A^1 = 0$, and $A^{1'} = 0$.

Then from (2) we have

$$\sum_{k=2}^{p} \beta_1^{1k} \frac{A^k I^1}{N_1} + \sum_{k=2}^{p} \alpha_1^{jk} \frac{A^k D^1}{N_1} = 0$$
$$\sum_{k=2}^{p} \left(\beta_1^{1k} \frac{I^1}{N_1} + \alpha_1^{1k} \frac{D^1}{N_1} \right) A^k = 0$$

By the positivity of $\frac{I^1}{N_1}$, $\frac{D^1}{N_1}$, β_1^{1k} and α_1^{1k} for all k, we can conclude that $A^k = 0$ for k. □

Remark 1. *From the previous theorem we can conclude that if one zone k is not at the agree-free equilibrium, which means that if $A^{k'} \neq 0$, thus all the other zones can not be at the agree-free equilibrium. In other word, if $A^k \neq 0$ for some k, thus $A^{j'} > 0$ for all j and therefore A^j is an increasing function even if $A^j(0) = 0$. See the example in the Fig.(2).*

Theorem 2. *Suppose that system (1)-(3) is at a disagree-free equilibrium for some zone j. Then the model is at the disagree free equilibrium i.e. $D^k = 0$ for each zone k.*

Proof. As previously, for simplicity suppose that $j = 1$, i.e. the zone 1 is at the disagree-free equilibrium and there is no disagree opinion in the zone 1, thus $D^1 = 0$, and $D^{1'} = 0$.

Then from (3) we have

$$\sum_{k=2}^{p} \beta_2^{1k} \frac{D^k I^1}{N_1} + \sum_{k=2}^{p} \alpha_2^{1k} \frac{A^1 D^k}{N_1} = 0$$
$$\sum_{k=2}^{p} \left(\beta_2^{1k} \frac{I^1}{N_1} + \alpha_2^{1k} \frac{A^1}{N_1} \right) D^k = 0$$

By the positivity of $\frac{I^1}{N_1}$, $\frac{A^1}{N_1}$, β_2^{1k} and α_2^{1k} for all k, we can conclude that $D^k = 0$ for k. □

Remark 2. *From the previous theorem also we can see that if one zone k is not at the disagree-free equilibrium, thus all the other zones can not be at the disagree-free equilibrium, see the example presented in the Fig.(3).*

Theorem 3. *If the model (1)-(3) at the disagree-free equilibrium, that is $D^j = 0$ for all j, then if for some zone k such that N_k is sufficiently big we have A^k decreases towards 0.*

Proof. We have $D^j = 0$ for all j, and from (2) we have

$$A^{j'} = \sum_{k=1}^{p} \beta_1^{jk} \frac{A^k I^j}{N_j} - \gamma_1^j A^j$$

thus for N_j sufficiently big we get $A^{j'} < 0$, which completes the proof. □

The example presented in the Fig.(3) shows the disagree-free equilibrium of the model (1)-(3), and for zones 3 and 4, we can see that A^3 and A^4 decrease towards 0 because of the big values of N_3 and N_4, as described in theorem (3).

Theorem 4. *If the model (1)-(3) at the agree-free equilibrium, that is $A^j = 0$ for all j, then if for some zone k such that N_k is sufficiently big , D^k decreases towards 0.*

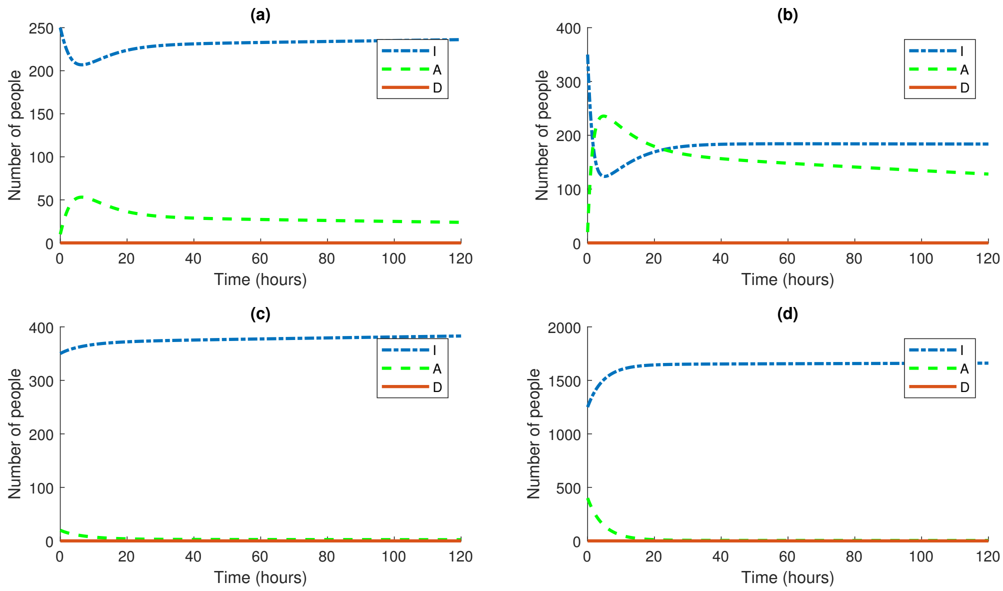

Figure 3: Example of the disagree-free equilibrium discussed in Theorems (2) and (3).

Table 2: Parameters' values and initial states

	$i=1$				$i=2$				$i=3$				$i=4$			
	$j=1$	$j=2$	$j=3$	$j=4$	$j=1$	$j=2$	$j=3$	$j=4$	$j=1$	$j=2$	$j=3$	$j=4$	$j=1$	$j=2$	$j=3$	$j=4$
β_1^{ij}	0.041	0.043	0.039	0.044	0.40	0.41	0.33	0.39	0.0043	0.0028	0.0036	0.0038	0.0041	0.0044	0.0039	0.0046
β_2^{ij}	0.2316	0.2826	0.3826	0.1816	0.005916	0.005810	0.006015	0.005956	0.005816	0.06116	0.05618	0.05419	0.005111	0.05880	0.04918	0.05826
α_1^{ij}	0.3346	0.3444	0.3116	0.3486	0.3946	0.3747	0.3226	0.3111	0.3340	0.3144	0.2946	0.3040	0.3915	0.3113	0.3006	0.3161
α_2^{ij}	0.2459	0.2258	0.2479	0.2549	0.2769	0.1959	0.2709	0.2249	0.2339	0.2444	0.2151	0.2009	0.1989	0.2069	0.2229	0.2141
γ_1^i	0.2579				0.2489				0.2629				0.2171			
γ_2^i	0.0938				0.0944				0.0929				0.0937			

Initial states

Zone	C_1			C_2			C_3			C_4		
Parameter	$I^1(0)$	$A^1(0)$	$D^1(0)$	$I^2(0)$	$A^2(0)$	$D^2(0)$	$I^3(0)$	$A^3(0)$	$D^3(0)$	$I^4(0)$	$A^4(0)$	$D^4(0)$
Data used in Fig.2	250	0	10	350	0	20	150	0	2	1250	0	400
Data used in Fig.3	250	10	0	350	20	0	350	20	0	1250	400	0
Data used in numerical simulation	250	20	30	350	20	40	350	30	30	1250	100	120

Proof. We have $A^j = 0$ for all j, and from (3) we have

$$D^{j\prime} = \sum_{k=1}^{p} \beta_2^{jk} \frac{D^k I^j}{N_j} - \gamma_2^j D^j$$

thus for N_j sufficiently big we get $D^{j\prime} < 0$, which completes the proof. □

The example presented in the Fig.(2) shows the agree-free equilibrium of the model (1)-(3), and for the zone 4, we can see that D^4 decreases towards 0 because of the big values of N_4, as described in theorem (4).

3 Optimal control problem

3.1 Presentation of the model with controls

Opinion polls have an important role in current political campaigns. Party leaders use opinion polls in the election campaign to mobilize voters and refine their campaign strategies. Because updates on party performance attract a lot of media attention and often serve as the basis for political commentary in the weeks leading up to election day. It is well known that learning the attitudes of electorates can shape the behavior of citizens, and it has often been criticized for its effects on their perceptions Ginsberg (1986); Herbst (1995). This explains why more than thirty democracies around the world have banned the publication of opinion polls in the run-up to election day at the turn of the century Pereira (2019).

The success of an election campaign depends to a large extent on its ability to use and formulate new information to its advantage, and the way party leaders respond to opinion polls is one of the cornerstones of this process. For example, Biden leads Trump by 10 percentage points in Wisconsin and Michigan, and the presidential nominee is ahead by seven points in Pennsylvania. Biden has led Trump in all three states in every Reuters/Ipsos weekly poll that began in mid-September 2020, and his leads have ticked higher in each state over the two weeks leading to the electoral day Reuters polling system (2020b). Due to the difficulty of controlling all regions, in some situations, as in the presidential election of 2020, it could be crucial to determine the states that will create the difference and then target them by control interventions to bring the situation under control.

As an application of the results in Bidah et al. (2020a), we investigate also here the impact of media programs and publicity in changing people's opinions during opinion polls, but this time, in a multi-zone framework. To do this, we introduce the two control variables u_1^j that represents the effect of publicity and positive media programs to attract more people in the positive opinion group in the zone C_j, based on real facts and providing people with more accurate and realistic information in an easy way that all people use such as WhatsApp, Facebook, Tweeter,... Thus this control targets the Indifferent group of C_j to bring them to the agreeing group of C_j, that is, an Indifferent individual becomes agreeing at a rate $u_1^j I^j$.

And the second control u_2^j that represents the effect of negative media programs against competitors. This control targets the disagreeing and abstaining people to change their mind, by providing them with negative information about the competitor or information clarifying certain ambiguities to at least motivate them not to abstain. For instance, partisans shift their opinions away from their party's positions when policy information provides a compelling reason for doing so Boudreau & MacKenzie (2014). Thus a disagreeing individual becomes again Indifferent of C_j at a rate $u_2^j D^j$. Therefore, the controlled model for the targeted zone C_j takes the following form

$$I^{j\prime} = -\sum_{k=1}^{p} \beta_1^{jk} \frac{A^k I^j}{N_j} - \sum_{k=1}^{p} \beta_2^{jk} \frac{D^k I^j}{N_j} + \gamma_1^j A^j + \gamma_2^j D^j - u_1^j I^j + u_2^j D^j \tag{4}$$

$$A^{j\prime} = \sum_{k=1}^{p} \beta_1^{jk} \frac{A^k I^j}{N_j} + \sum_{k=1}^{p} \alpha_1^{jk} \frac{A^k D^j}{N_j} - \sum_{k=1}^{p} \alpha_2^{jk} \frac{A^j D^k}{N_j} - \gamma_1^j A^j + u_1^j I^j \tag{5}$$

$$D^{j\prime} = \sum_{k=1}^{p} \beta_2^{jk} \frac{D^k I^j}{N_j} + \sum_{k=1}^{p} \alpha_2^{jk} \frac{A^j D^k}{N_j} - \sum_{k=1}^{p} \alpha_1^{jk} \frac{A^k D^j}{N_j} - \gamma_2^j D^j - u_2^j D^j \tag{6}$$

Where $I^j(0) \geq 0$, $A^j(0) \geq 0$, and $D^j(0) \geq 0$.

3.2 Optimal control problem

Now, we consider an optimal control problem to minimize the objective functional

$$J(u_1^j, u_2^j) = \int_0^{t_f} \left(c_1 I^j(t) - c_2 A^j(t) + c_3 D^j(t) + \frac{K_1}{2} \left(u_1^j(t)\right)^2 + \frac{K_2}{2} \left(u_2^j(t)\right)^2 \right) dt$$

where c_1, c_2 and c_2 are small positive constants to keep a balance in the size of $I^j(t)$, $A^j(t)$ and $D^j(t)$, respectively. The positive constants K_1 and K_2 balance the size of quadratic control terms. The reason behind considering a finite time horizon is that the control period is usually restricted to a limited time window. The objective of our work here is to minimize the Indifferent and Disagree groups by using possible minimal costs of applying control variables $u_1^j(t)$ and $u_2^j(t)$ attempting to increase the number of the Agreeing people.

We seek an optimal control pair $\left(u_1^{j*}, u_2^{j*}\right)$ such that

$$J(u_1^{j*}, u_2^{j*}) = min\left\{J(u_1^j, u_2^j)|(u_1^j, u_2^j) \in U\right\} \tag{7}$$

subject to (4)-(6).
Where

$$U = \left\{(u_1^j, u_2^j)|u_1^j, u_2^j \text{ measurables}, 0 \leq u_1^j \leq 1,\ 0 \leq u_2^j \leq 1,\ t \in [0, t_f]\right\} \tag{8}$$

In order to find an optimal solution, first we find the Lagrangian and Hamiltonian for our optimal control problem. In fact, the Lagrangian of the optimal problem is given by

$$\mathcal{L}\left(I^j, A^j, D^j, u_1^j, u_2^j\right) = c_1 I^j(t) - c_2 A^j(t) + c_3 D^j(t) + \frac{K_1}{2}\left(u_1^j(t)\right)^2 + \frac{K_2}{2}\left(u_2^j(t)\right)^2$$

3.3 Existence of an optimal solution

To prove that there is an optimal solution of problem (7), we will use a result, Theorem 5 below, that ensures the existence of the solution for optimal control problems contained in Theorem III.4.1 and Corollary III.4.1 in Fleming & Risheln (2012). Problem (7) is an optimal control problem in Lagrange form:

$$\begin{gathered} J(x, u) = \int_{t_0}^{t_1} \mathcal{L}(t, x(t), u(t))\, dt \rightarrow min \\ \begin{cases} x'(t) = & f(t, x(t), u(t)) \\ x(t_0) = & x_0, \end{cases} \\ x(.) \in AC([t_0, t_1]; \mathbb{R}^n),\ u(.) \in L^1([t_0, t_1]; U \in \mathbb{R}^m) \end{gathered} \tag{9}$$

Where $AC([t_0, t_1]; \mathbb{R}^n)$ is a space of absolutely continuous functions defined on the closed interval $[t_0, t_1]$ with values in $\mathbb{R}^n$. In the above context, we say that a pair $(x, u) \in AC([t_0, t_1]; \mathbb{R}^n) \times L^1([t_0, t_1]; U \in \mathbb{R}^m)$ is feasible if it satisfies the Cauchy problem in (9) . We denote the set of all feasible pairs by $\mathcal{F}$. Next, we recall

Theorem 5. *(See Fleming & Risheln (2012)) For problem (7), suppose that f and $\mathcal{L}$ are continuous and there exist positive constants C_1 and C_2 such that, for $t \in \mathbb{R}$, x, x_1 , $x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, we have*

1) $\|f(t, x(t), u(t))\| \leq C_1(1 + \|x\| + \|u\|)$

2) $\|f(t, x_1(t), u(t)) - f(t, x_2(t), u(t))\| \leq C_2\|x_1 - x_2\|(1 + \|u\|)$

3) $\mathcal{F}$ is non empty set.

4) U is closed.

5) There is a compact set S such that $x(t_1) \in S$ for any state variable x.

6) U is convex, $f(t, x, u) = \alpha(t, x) + \beta(t, x)u$, and $\mathcal{L}(t, x, .)$ is convex on U.

7) $\mathcal{L}(t, x, u) \geq c_1|u|^\beta - c_2$, for some $c_1 > 0$ and $\beta > 1$.

Then, there exist (x^*, u^*) minimizing J on $\mathcal{F}$.

Applying Theorem 5 to our problem we obtain the following result:

Theorem 6. *There exists an optimal control pair (u_1^*, u_2^*) and a corresponding solution of the initial value problem in (7), (I^*, A^*, D^*), that minimizes the cost functional J in (7) over $L^1([0, t_f]; [0, 1] \times [0, 1]))$.*

Proof. We first note that, adding the equations in (4)-(6), we conclude that the total population is constant for each zone: $N_j(t) = I^j(0) + A^j(0) + D^j(0) = N_j(0)$. Thus

$$I^j(t),\ A^j(t), D^j(t) \leq N_j(0)$$

Additionally, $\frac{A^k(t)I^j(t)}{N_j(t)} \leq A^k(t) \leq \max_{k=1,\ldots,p}(N_k)$, $\frac{A^j(t)D^k(t)}{N_j(t)} \leq D^k(t) \leq \max_{k=1,\ldots,p}(N_k)$ and $\frac{D^j(t)I^k(t)}{N_j(t)} \leq I^k(t) \leq \max_{k=1,\ldots,p}(N_k)$. We immediately obtain 1) and 2).

Conditions 3) and 4) are immediate from the definition of $\mathcal{F}$ since $U = [0,1] \times [0,1]$. We conclude that all the state variables are in the compact set

$$\left\{(x,y,z) \in \left(\mathbb{R}^+\right)^3 : 0 \leq x+y+z \leq \max_{k=1,\ldots,p}(N_k)\right\}$$

and condition 5) follows. Since the state equations are linearly dependent on the controls and $\mathcal{L}$ is quadratic in the controls, we obtain 6). Finally,

$$\begin{aligned}\mathcal{L} &= c_1 I^j(t) - c_2 A^j(t) + c_3 D^j(t) + \frac{K_1}{2}\left(u_1^j(t)\right)^2 + \frac{K_2}{2}\left(u_2^j(t)\right)^2 \\ &\geq \min\left\{\frac{K_1}{2}, \frac{K_2}{2}\right\}\left\|\left(u_1^j, u_2^j\right)\right\|^2\end{aligned}$$

and we establish 7) with $c_1 = \min\left\{\frac{K_1}{2}, \frac{K_2}{2}\right\}$. Therefore the result follows from Theorem 5. $\square$

3.4 Necessary conditions of optimality

We seek the minimal value of the Lagrangian. To accomplish this, we define the Hamiltonian $\mathcal{H}$ as follows

$$\begin{aligned}\mathcal{H} &= \mathcal{L}\left(I^j, A^j, D^j, u_1^j, u_2^j\right) \\ &+ \sum_{j=1}^{p}\left[\lambda_1^j(t)\left[-\sum_{k=1}^{p}\beta_1^{jk}\frac{A^k I^j}{N_j} - \sum_{k=1}^{p}\beta_2^{jk}\frac{D^k I^j}{N_j} + \gamma_1^j A^j + \gamma_2^j D^j - u_1^j I^j + u_2^j D^j\right]\right. \\ &+ \lambda_2^j(t)\left[\sum_{k=1}^{p}\beta_1^{jk}\frac{A^k I^j}{N_j} + \sum_{k=1}^{p}\alpha_1^{jk}\frac{A^k D^j}{N_j} - \sum_{k=1}^{p}\alpha_2^{jk}\frac{A^j D^k}{N_j} - \gamma_1^j A^j + u_1^j I^j\right] \\ &+ \left.\lambda_3^j(t)\left[\sum_{k=1}^{p}\beta_2^{jk}\frac{D^k I^j}{N_j} + \sum_{k=1}^{p}\alpha_2^{jk}\frac{A^j D^k}{N_j} - \sum_{k=1}^{p}\alpha_1^{jk}\frac{A^k D^j}{N_j} - \gamma_2^j D^j - u_2^j D^j\right]\right]\end{aligned} \tag{10}$$

To find the optimal solution, we apply the Pontryagin's Maximum Principle to the Hamiltonian Pontryagin et al. (1962), and we obtain the following theorem.

Theorem 7. *Let $I^{j*}(t)$, $A^{j*}(t)$ and $D^{j*}(t)$ be optimal state solutions with associated optimal control variables $u_1^{j*}(t)$ and $u_2^{j*}(t)$ for the optimal control problem (7). Then, there exist adjoint* *variables $\lambda_1^j(t)$, $\lambda_2^j(t)$ and $\lambda_3^j(t)$ that satisfy*

$$\begin{aligned}
\lambda_1^{j\prime}(t) &= -\left[c_1 + \left(\lambda_2^j(t) - \lambda_1^j(t)\right)\left(u_1^j(t) + \sum_{k=1}^{p} \beta_1^{jk}\frac{A^k}{N_j}\right)\right. \\
&+ \left.\left(\lambda_3^j(t) - \lambda_1^j(t)\right)\sum_{k=1}^{p}\beta_2^{jk}\frac{D^k}{N_j}\right] \\
\lambda_2^{j\prime}(t) &= -\left[-c_2 + \sum_{k=1}^{p}\left(\lambda_2^k(t) - \lambda_1^k(t)\right)\frac{I^k(t)\,\beta_1^{kj}}{N_k} + \sum_{k=1}^{p}\left(\lambda_2^k(t) - \lambda_3^k(t)\right)\frac{D^k\alpha_1^{kj}}{N_k}\right. \\
&+ \left.\left(\lambda_1^j(t) - \lambda_2^j(t)\right)\gamma_1^j + \left(\lambda_3^j(t) - \lambda_2^j(t)\right)\sum_{k=1}^{p}\alpha_2^{jk}\frac{D^k}{N_j}\right] \\
\lambda_3^{j\prime}(t) &= -\left[c_3 + \sum_{k=1}^{p}\left(\lambda_3^k(t) - \lambda_1^k(t)\right)\frac{I^k\beta_2^{kj}}{N_k} + \sum_{k=1}^{p}\left(\lambda_3^k(t) - \lambda_2^k(t)\right)\frac{A^k\alpha_2^{kj}}{N_k}\right. \\
&+ \left.\left(\lambda_2^j(t) - \lambda_3^j(t)\right)\sum_{k=1}^{p}\alpha_1^{jk}\frac{A^k}{N_j} + \left(\lambda_1^j(t) - \lambda_3^j(t)\right)\left(\gamma_2^j + u_2^j(t)\right)\right]
\end{aligned}$$

with the transversality conditions $\lambda_i(t_f) = 0$, $i = 1, 2, 3$. Furthermore, the optimal controls $u_1^{j*}(t)$ and $u_2^{j*}(t)$ are given by

$$\begin{aligned}
u_1^*(t) &= max\left\{min\left\{\frac{I^j(t)\,\left(\lambda_1^j(t) - \lambda_2^j(t)\right)}{K_1}, 1\right\}, 0\right\} \\
u_2^*(t) &= max\left\{min\left\{\frac{D^j(t)\,\left(\lambda_3^j(t) - \lambda_1^j(t)\right)}{K_2}, 1\right\}, 0\right\}
\end{aligned}$$

Proof. To determine the adjoint equations and the transversality conditions, we use the Hamiltonian $\mathcal{H}$ defined by (10). From setting $I^j(t) = I^{j*}(t)$, $A^j(t) = A^{j*}(t)$ and $D^j(t) = D^{j*}(t)$, and differentiating $\mathcal{H}$ with respect to $I^j(t)$, $A^j(t)$ and $D^j(t)$, we obtain

$$\begin{aligned}
\lambda_1^{j\prime}(t) &= -\frac{\partial\mathcal{H}}{\partial I^j} \\
&= -\left[c_1 - \lambda_1^j(t)\left(u_1^j(t) + \sum_{k=1}^{p}\beta_1^{jk}\frac{A^k}{N_j} + \sum_{k=1}^{p}\beta_2^{jk}\frac{D^k}{N_j}\right)\right. \\
&+ \left.\lambda_2^j(t)\left(u_1^j(t) + \sum_{k=1}^{p}\beta_1^{jk}\frac{A^k}{N_j}\right) + \sum_{k=1}^{p}\beta_2^{jk}\frac{D^k\lambda_3^j(t)}{N_j}\right] \\
\lambda_2^{j\prime}(t) &= -\frac{\partial\mathcal{H}}{\partial A^j} \\
&= -\left[-c_2 + \sum_{k=1}^{p}\left(\lambda_2^k(t) - \lambda_1^k(t)\right)\frac{I^k(t)\,\beta_1^{kj}}{N_k} + \sum_{k=1}^{p}\left(\lambda_2^k(t) - \lambda_3^k(t)\right)\frac{D^k\alpha_1^{kj}}{N_k}\right. \\
&+ \left.\left(\lambda_1^j(t) - \lambda_2^j(t)\right)\gamma_1^j + \left(\lambda_3^j(t) - \lambda_2^j(t)\right)\sum_{k=1}^{p}\alpha_2^{jk}\frac{D^k}{N_j}\right] \\
\lambda_3^{j\prime}(t) &= -\frac{\partial\mathcal{H}}{\partial D^j} \\
&= -\left[c_3 + \sum_{k=1}^{p}\left(\lambda_3^k(t) - \lambda_1^k(t)\right)\frac{I^k\beta_2^{kj}}{N_k} + \sum_{k=1}^{p}\left(\lambda_3^k(t) - \lambda_2^k(t)\right)\frac{A^k\alpha_2^{kj}}{N_k}\right. \\
&+ \left.\left(\lambda_2^j(t) - \lambda_3^j(t)\right)\sum_{k=1}^{p}\alpha_1^{jk}\frac{A^k}{N_j} + \left(\lambda_1^j(t) - \lambda_3^j(t)\right)\left(\gamma_2^j + u_2^j(t)\right)\right]
\end{aligned}$$

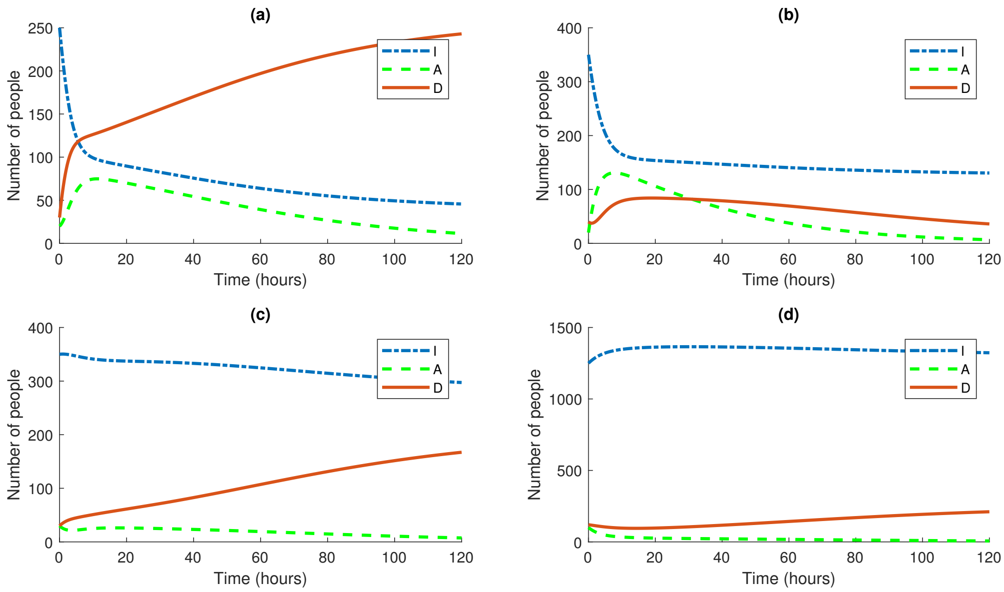

Figure 4: State variables of the model (1)-(3) without controls.

By the optimality conditions, we have

$$\frac{\partial \mathcal{H}}{\partial u_1^j} = I^j\,\lambda_2^j - I^j\,\lambda_1^j + K_1\,u_1^j \quad = \quad 0$$

then

$$u_1^j(t) = \frac{I^j\,(t)\,\left(\lambda_1^j\,(t) - \lambda_2^j\,(t)\right)}{K_1}$$

and from

$$\frac{\partial \mathcal{H}}{\partial u_2^j} = D^j\,\lambda_1^j - D^j\,\lambda_3^j + K_2\,u_2^j \quad = \quad 0$$

we have

$$u_2^j(t) = \frac{D^j\,(t)\,\left(\lambda_3^j\,(t) - \lambda_1^j\,(t)\right)}{K_2}$$

As our controls are bounded below by 0 and above by 1, thus we have

$$\begin{aligned} u_1^*(t) &= max\left\{min\left\{\frac{I^j\,(t)\,\left(\lambda_1^j\,(t) - \lambda_2^j\,(t)\right)}{K_1}, 1\right\}, 0\right\} \\ u_2^*(t) &= max\left\{min\left\{\frac{D^j\,(t)\,\left(\lambda_3^j\,(t) - \lambda_1^j\,(t)\right)}{K_2}, 1\right\}, 0\right\} \end{aligned}$$

□

4 Numerical simulation

We now present numerical simulations associated with our optimal system derived from the previous mathematical model. We wrote a code in MATLAB^{TM} and simulated our results using different data. We solve the optimality systems using an iterative method with a progressive-regressive Runge–Kutta fourth-order scheme. Such numerical procedures are called forward–backward sweep methods, where the state system with an initial guess is solved forward in time and then the adjoint system is solved backward in time. First, starting with an initial guess for the adjoint variables, we solve the state equations by a forward Runge–Kutta fourth-order procedure in time.

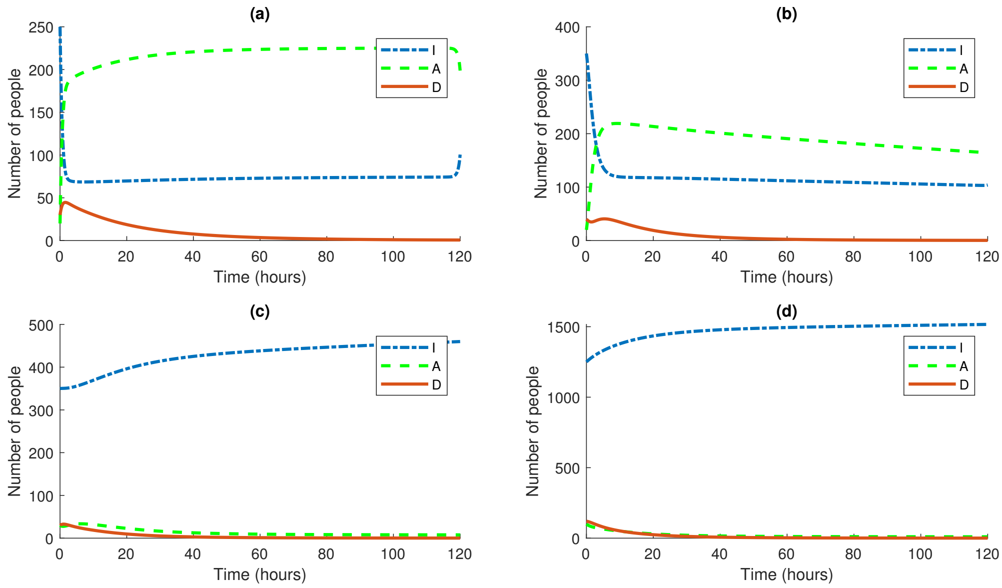

Figure 5: State variables of the model (4)-(6) with controls.

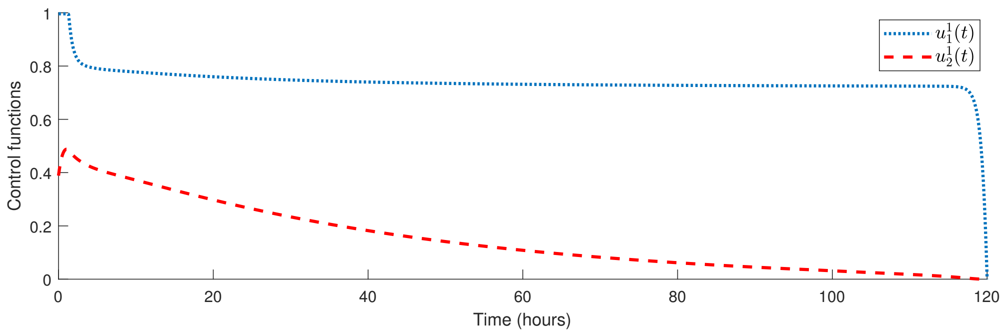

Figure 6: Control variables

Then, those state values are used to solve the adjoint equations by a backward Runge–Kutta fourth order procedure because of the transversality conditions Zakary et al. (2016); **?**); Jung et al. (2002); Lenhart & Workman (2013). Afterwards, we updated the optimal control values using the values of state and co-state variables obtained in the previous steps. Finally, we execute the previous steps until a tolerance criterion is reached.

4.1 Scenario 1

In the following simulations, we use parameters' values given in Table 2. We chose these parameters here because we know the poll result in all zones, and in such situation, the control intervention is needed, as it can be seen in Fig.4. The interrogation period is 5 days and users are allowed to change their minds.

Fig.4 depicts the state variables I^j, A^j, and D^j for $j = 1,..,4$, of the model (1)-(3) when there is no control intervention in the four considered zones.

It can be seen that from the beginning of the poll until about 15 hours, the number of people approving in the zone C_1 increases significantly, then begins to decrease continuously until the end of the survey. While the number of people who disagree increases rapidly by about 15 hours, and it continues to increase until the end of the survey. The number of Indifferent people decreases very rapidly in the first 15 hours, from 250 to about 100, to begin decreases slightly.

In the zone C_2 we can see that the number of approving people increases rapidly to reach its peak at 130 individuals then it begins to decrease towards zero by the end of the poll. While the number of disapproving people retains small values compared to the number of approving individuals, but by about 30 hours it can be seen that this number exceeds the approving people number. The number of indifferent individuals decreases also quickly in the first 10 hours from 350 to 170 individuals, to continue decreasing slightly.

In the zone C_3, the course of events is very clear, while it can be seen that from the beginning of the poll, the number of people approving decreases towards zero and the number of disapproving people increases continuously until the end of the investigation. The number of indifferent people decreases slightly from the beginning of the poll.

Also in the zone C_4 we can see the result of the poll from its beginning, where it can be seen that the number of approving individuals tends to zero while the number of disagreeing people increases slightly until the end. the number of indifferent people converge to a big value.

Around 30 hours, we can predict the result of this survey. Where the number of people in disagreement continues to increase and the number of people in agreement continues to decrease. However, implementing control strategies may take some time to influence the outcome, and therefore it is recommended that controls be introduced early.

Fig.5 depicts the state variables of the model (4)-(6) when controls are applied from the beginning of the survey in the zone C_1. It can be seen that the number of people who approved in the zone C_1 increased and stabilized at around 210 individuals until the end of the survey. And the number of Indifferent people decreased and stabilized at around 75 people until the end of the survey. While the number of disapproving people tends to zero from the beginning of the control strategy. This control strategy gives satisfactory results within 5 hours.

In C_2 we can see that the number of approving people increases quickly to reach its peak around 210 individuals within the first 10 hours, then it begins to decrease slightly towards 190 individuals at the end of the poll. The number of indifferent people decreases within the first 10 hours to stabilize around 110 individuals. While the number of disapproving people decreases towards zero.

In C_3 it can be seen that the control strategy inverted the course of events, while we can see that number of disapproving people tends to zero from the beginning of the poll to take small values compared to case when there are no controls, and the number of approving individuals exceeds the one of disapproving people even it takes small values. Also in the zone C_4 we can see that this strategy of control can reduce the number of disapproving people by making its number takes small values compared to case when there are no controls where this number continues to grow continuously, and the number of approving people exceeds the one of disapproving individuals even it takes small values.

Note that the control function u_1^1 decrease quickly at first 5 hours and settle at constant values around 0.8 until the end of the interrogation time, see Fig.6. The u_2^1 control reaches its peak about 0.5 within the first 3 hours then it begins to decrease slightly till the end of the poll. This simulation shows the effectiveness of optimal controls to reduce the number of people who disapprove and increase the number of approving people in the targeted zone.

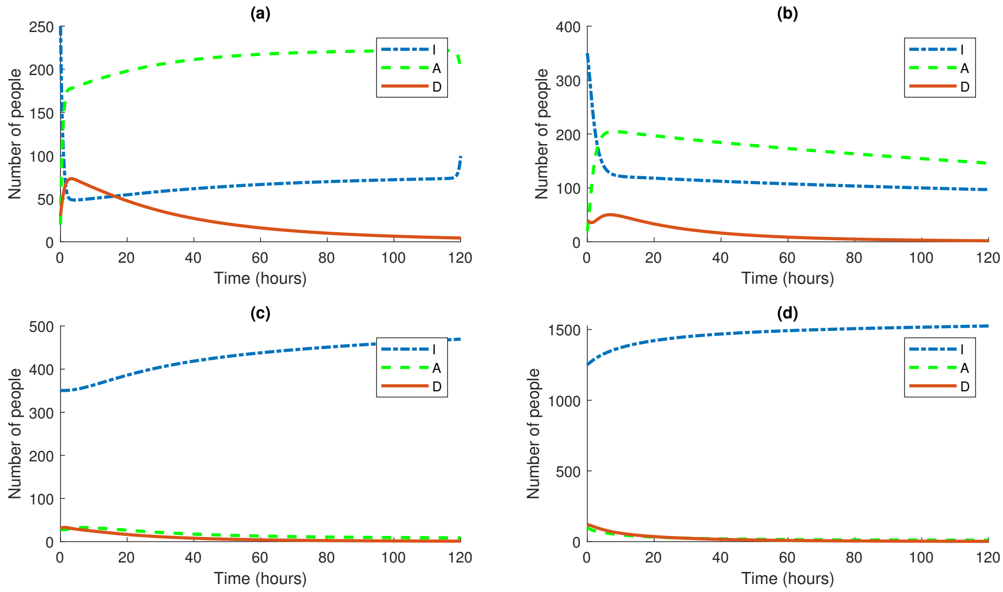

Figure 7: State variables of the model (4)-(6) with controls. Only the control u_1^1 is used.

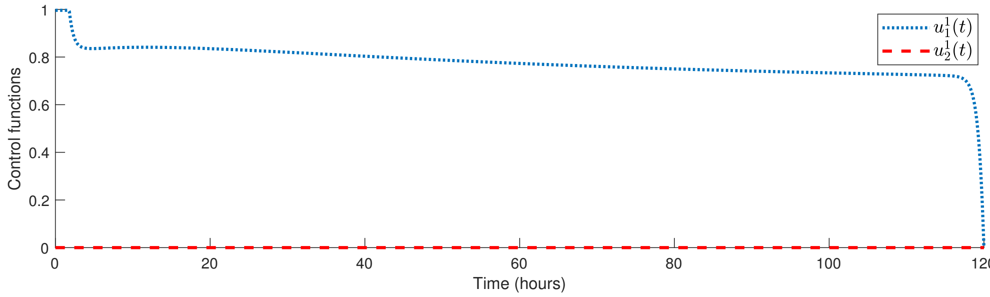

Figure 8: Control variables in the case of using u_1^1 only

4.2 Scenario 2

As a different scenario, we discuss hereafter the use of one control in the targeted zone instead of two controls. Fig.7 depicts the state variables of the model (4)-(6) when only the control u_1^1 is used. It can be seen that this strategy of control is also effective in reducing the number of disapproving people and increasing the number of approving individuals by about 5 hours in zones C_1 and C_2. While it makes the disapproving people tend to zero in C_3 and C_4.

It can be seen that the number of people approving increases to about 210 individuals. While the number of Indifferent people decreases quickly within the first 5 hours to around 50 individuals and then begin to increase towards 100 individuals by the end of the survey. Note that the number of disapproved people reaches its peak around 75 individuals and then it begins to decrease towards zero.

We can see that the control variable u_1^1 starts from its maximum value of 1 until about 5 hours to begin decreasing to around 0.9 and it continues decreasing slightly until the end of the control strategy, see Fig.8.

In the Fig.9 we can see the state variables of the model (4)-(6) when only the control u_2^1 is used. We can see that the number of people approving in C_1 increases to about 75 individuals and then begins to decrease until the end of the poll. While the number of Indifferent people decreases to 110 individuals and then it continues decreasing slightly until the end of the survey. The number of disapproved people increases within the first 5 hours to around 110 individuals and it continues increasing till the end. Regarding the other zones, in overall, it seems that this strategy of control is not sufficient even if the control maintains a constant value until the end of the poll, see Fig.10.

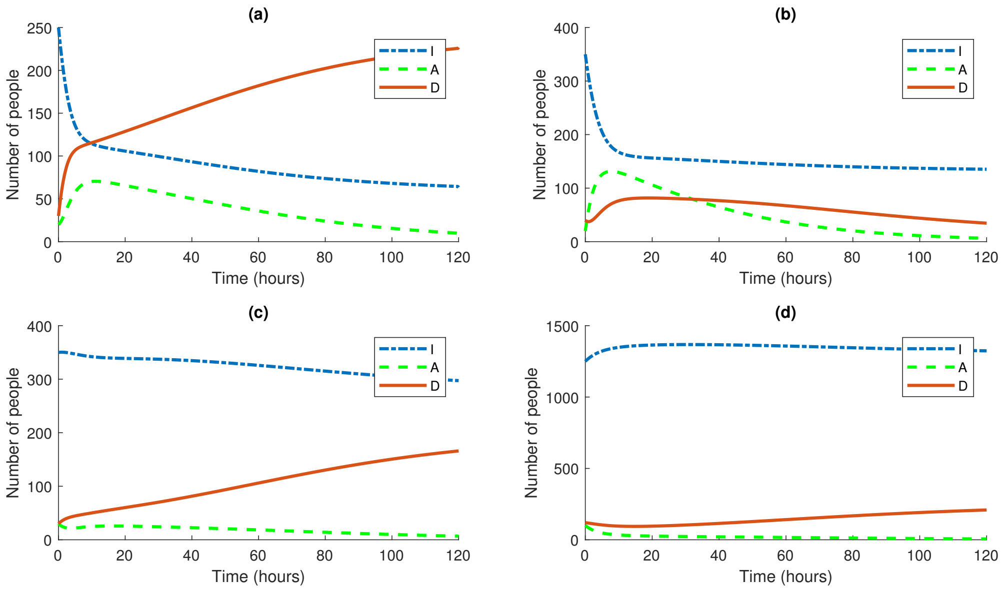

Figure 9: State variables of the model (4)-(6) with controls. Only the control u_2^1 is used.

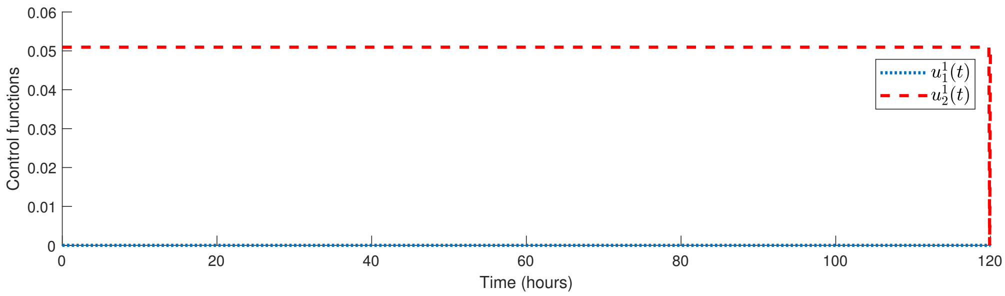

Figure 10: Control variables in the case of using u_2^1 only

5 Conclusion

In this paper, we applied optimal control to a generalized mathematical model that describes the evolution of opinions during polls in several geographic zones. We presented some properties of the multi-zone model illustrated by different examples. We incorporated two control variables in a targeted zone, the first presents the effects of the media and publicity to convince people to change their mind and then bring them to the agreeing group. While the second control is the effects of negative media against competitors by providing people with negative information about the competitor or information clarifying certain ambiguities to at least motivate them not to vote. We proved the existence of optimal controls that ensure the minimization of Indifferent and disagreeing individuals by using possible minimal costs of control application. We characterized optimal controls by using Pontryagin's Maximum Principle and developed the optimality system and solved it using an iterative numerical method in order to simulate several possible scenarios, with and without optimal controls. We found that targeting only one zone by controls can effectively reduce disagreeing individuals in all the other zones. Furthermore, the most effective control strategy is the use of the two proposed controls in the targeted zone or at least one control of positive media which targets the indifferent people. While using only negative media remains insufficient to bring the situation under control.

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